Unsolved Matrix Problems

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Let $\rho(\cdot)$ denote the spectral radius, i.e., the maximum modulus of eigenvalues of a matrix. Let $\circ$ denote the Hadamard product, i.e., entry-wise product of matrices.

Conjecture 1. *If A, B are nonnegative matrices of the same order, then*

$$\rho(A \circ B) \leq \rho(AB).$$

Remark Sept 7,2010: Conjecture 1 has been proved by K.M.R. Audenaert, Spectral radius of Hadamard product versus conventional product for non-negative matrices, Linear Algebra Appl., 432 (2010) 366-368 and by R.A. Horn and F. Zhang, Bounds on the spectral radius of a Hadamard product of nonnegative or positive semidefinite matrices, Electron. J. Linear Algebra, 20 (2010) 90-94 respectively. Z. Huang has extended this inequality to the case of 3 or more nonnegative matrices in his paper On the spectral radius and the spectral norm of Hadamard products of nonnegative matrices, to appear in LAA.

Let $Z(n)$ denote the set of 0-1 matrices of order n. Let $f(A)$ denote the number of $1's$ in a matrix A. For positive integers n, k let

$$\gamma(n,k) = \max\{f(A)|A \in Z(n),\ A^k \in Z(n)\},$$

$$\beta(n,k) = \max\{f(A^k)|A \in Z(n)\}.$$

In 2007 I posed the following two problems at a seminar.

Problem 2. *Determine $\gamma(n,k)$ and determine the 0-1 matrices that attain this maximum number.*
The case $k = 2$ is solved by Wu [7], and the case $k \geq n-1$ is solved by Huang and Zhan [5].

Let $A = (a_{ij}) \in Z(n)$. The digraph of A is the digraph with vertices $1, \ldots, n$ and in which (i,j) is an arc if and only if $a_{ij} \neq 0$. This gives a bijection between $Z(n)$ and the set of digraphs of order n.

The graph-theoretic interpretation of Problem 2:
Let D be a digraph of order n. If for each pair of vertices i, j of D there is at most one walk of length k from i to j, then what is the maximum number of arcs in D? Determine the digraphs that attain this maximum number.

Problem 3. *Determine $\beta(n,k)$ and determine the 0-1 matrices that attain this maximum number. We may also consider the analogous problem for symmetric 0-1 matrices.*

The graph-theoretic interpretation of Problem 3:
Let D be a digraph of order n. What is the maximum number of pairs of vertices i, j of D for which there is exactly one walk of length k from i to j? Determine the digraphs that attain this maximum number.

It is known [2, 6] that $\beta(n,k) = n^2$ if and only if there is a positive integer m such that $m^k = n$.
We can see that

$$\beta(n,2) = \begin{cases} n^2 & \text{if } n \text{ is a square,} \\ n^2 - 1 & \text{otherwise.} \end{cases}$$

For a complex matrix A, denote the entry-wise complex conjugate, the transpose and the conjugate transpose of A by $\bar{A}$, A^T and A^* respectively.

In 2007 I posed the following conjecture at a seminar.

Conjecture 4. *Let A, B be complex matrices of the same order. Then*

$$\|(A \circ B)(A \circ B)^*\| \le \|(A \circ \bar{A})(B \circ \bar{B})^T\|$$

for any unitarily invariant norm $\|\cdot\|$.

Du [3] has proved the special cases when $\|\cdot\|$ is the spectral norm, the trace norm, or the Frobenius norm. See [1, 4] for properties of unitarily invariant norms.

Remark Sept 7, 2010: Conjecture 4 is false in general. Counterexamples are given in Z. Huang, On the spectral radius and the spectral norm of Hadamard products of nonnegative matrices, to appear in LAA.

Let $S_n[a, b]$ denote the set of $n \times n$ real symmetric matrices whose entries are in the interval $[a, b]$. For an $n \times n$ real symmetric matrix A, we denote the eigenvalues of A in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The *spread* of such an A is defined to be $s(A) = \lambda_1(A) - \lambda_n(A)$.

I posed the following two problems in [8] in 2006.

Problem 5. *For a given integer j with $2 \leq j \leq n-1$, determine*

$$\max\{\lambda_j(A) : A \in S_n[a, b]\},$$

$$\min\{\lambda_j(A) : A \in S_n[a, b]\}$$

and determine the matrices that attain the maximum and the matrices that attain the minimum.

The cases $j = 1, n$ are solved [8].

Problem 6. *For generic $a < b$ determine*

$$\mathsf{max}\{s(A) : A \in S_n[a,b]\}$$

and determine the matrices that attain the maximum.

The case $a = -b$ is solved [8].

Let $\phi(A)$ be the number of nonzero entries of a matrix A. At the 12th ILAS conference, Regina, Canada, 2005 I posed the following problem.

Problem 7. *Characterize the sign patterns of entry-wise nonnegative matrices A such that the sequence $\{\phi(A^k)\}_{k=1}^{\infty}$ is nondecreasing. We may consider the same problem with "nondecreasing" replaced by "nonincreasing".*

Remark: Sidak observed in 1964 that there exists a primitive matrix A of order 9 satisfying

$$18 = \phi(A) > \phi(A^2) = 16.$$

References

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Thank you!